**Continuous growth models:** Let  be the population of a species at time . Then the general form of a single species population model is



This is also called conservation equation for the population. The simplest model has no migration and the birth and death terms are proportional to . There are some continuous growth models such as:

1. **Malthusian model:** The Malthusian model is



whereis proportionality constant and .

1. **Logistic Model:** The Logistic model or Pearl-Verhulst model is



whereis carrying capacity and is a positive constant.

1. **Delay Model:** If the birth rate of a population is considered to act instantaneously whereas there maybe a time of delay to take account of the time to reach maturity, the finite gestation period and so on, then the delay model is



whereis delay parameter.

1. **Harvesting Model:** The harvesting model is



whereand  are positive constants,  is the linear growth rate,  is the natural carrying capacity and  is the harvesting.

**Carrying Capacity:** The carrying capacity of a biological [species](https://en.wikipedia.org/wiki/Species) in an [environment](https://en.wikipedia.org/wiki/Natural_environment) is the maximum population size of the species that the environment can sustain indefinitely, given the food, [habitat](https://en.wikipedia.org/wiki/Habitat), [water](https://en.wikipedia.org/wiki/Drinking_water), and other [necessities available](https://en.wikipedia.org/wiki/Resource) in the environment.

**Malthusian model:** We consider a population of single species. The growth rate of species is proportional to the number of species at any time . The proportionality constant may be dependent or independent of the number of species and time. Let be the number of population of the species. Then the first order linear ordinary differential equation

 

where,  is a constant and , is called the Malthusian model.

From (1), we get



Integrating, 

 

where is an integrating constant.

If at the initial population is  , then from (2) we get





Thus the solution of (1) is given by



Figure: Malthusian model

This represents the population at any time . It shows that the population grows exponentially if , decays exponentially if  and remains constant if .

**Stability:** If the model (1) is stable, if the model (1) is unstable and if the model (1) is

asymptotically stable.

Since  if 

if

if

The maximum population is



Here  is the equilibrium or critical point.

**Comment:** From the Malthusian model, we observe that the population increases exponentially with

time and  as . Thus the growth is unlimited. This model is remarkably accurate in the case of human population of the earth during the last several decades. It is also outstanding accurate for comparatively smaller size of the population under short period of time. But it is unrealistic when applied to distant future and completely unrealistic for large size of population used over sufficiently long period of time. For in reality after a certain period of time, the population attains a maximum size and remains constant.

**Q-01:** Describe the Malthusian model for the dynamics of a single species population and comment on its plausibility. Discuss the stability of the equilibrium states of the model.

**Logistic model:** We consider a population of single species and let be the number of population of the species at any time . Then the first order non-linear ordinary differential equation



 

where, is called the Logistic model or Pearl-Verhulst model. Here  is called carrying capacity,  and  are positive constants and  is very small compared to .

The equation (1) is a separable or Bernoulli type equation. So separating the variables, we get









Integrating, 

 

where is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get















 

This represents the population at any time . It is complete solution of the model.

The maximum population is





Figure: Logistic model

From the graph, we observe that if the population increases and approaches toas . Ifthe population decreases and approaches to as . The population remains constant when . Thus it is clear that  is a limiting factor or limiting behavior on the growth of the population.

**Stability:** The equilibrium or critical points or steady states of the logistic model are given by







and

For linearization about  we put  with  in (1). Thus we have



Since  so neglecting  term, we get







This shows that as . Thus is an unstable equilibrium point.

Again for linearization about  we put  with  in (1). Thus we have







Since  so neglecting  term, we get





This shows that as . Thus the equilibrium  i.e. is stable.

**Comment:** From the Logistic model, we observe that the population does not increase exponentially

with time and  as . Thus  is a limiting factor on the growth of the population. This model is remarkably accurate in the case of human population of the earth during the last several decades.

**Q-02:** Describe the Logistic model or Pearl-Verhulst model for the dynamics of a single species population and comment on its plausibility. Discuss the stability of the equilibrium states of the model.

**OR**

* Interpret the parameters  and  in the logistic growth model  for a single species population. Obtain a formula for the population size . Determine the steady states and discuss their stability.

**OR**

* Suppose that a certain population obeys the Verhulst model with intrinsic growth rate and carrying capacity . Find the complete solution of the model. Discuss also the limiting behavior of the model as .

**Delay model:** We consider a population of single species and let be the number of population of the species at any time . If the birth rate of the population is considered to act instantaneously where as there may be a time of delay to take account of the time to reach maturity, the finite gestation period and so on, then the delay model is

 

whereis delay parameter.

By using (1), the logistic model can be written as

 

where and  are positive constants.

Here  depends on population size at earlier time, , rather than that at . The effect of on  depends on and is measured by a positive weight function . The weighted average of the effects of all earlier population sizes on  is then given by



We shall always normalize the weight function



such that the influence of all earlier population sizes on  depends on



Thus a more accurate model than (2) is

 

Practically  will tend to zero for large negative and positive  and probably have a maximum at some representative time .

Figure-01: Schematic periodic solution of the delay population model.

Suppose that for some ,  and for some , . Then from (2) we have

,  and so  is still increasing at . When , , and so . If , , and so . In this case  decreases until  since then . Again, . Thus, there is the possibility of oscillatory behavior.

**Population growth with harvesting**: The removal of a population during each time period is known as harvesting which could be due to hunting, fishing or the spread of diseases. There has been a need to know how killing a certain number of animals will affect the population at large. To develop an ecologically acceptable strategy for harvesting any renewable resource, we usually want the maximal harvest with the minimum effort (sustainable level). There are two standard approaches to harvesting from a population. We can harvest a set number of individuals every time (**constant harvesting**), or we can harvest a set percentage of the population every time (**proportional harvesting**).

**Constant harvesting model:** We consider a population of single species and let be the number of population of the species at any time . If the harvesting occurs at a constant rate  then the constant harvesting model is defined as

 

where ,  and  are positive constants,  is the linear per capita growth rate,  is the natural carrying capacity and  is the constant rate of the population that is harvested in every unit of time.

The equation (1) is a separable equation. So separating the variables, we get











where , 



Integrating,  ; where 

 

where is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get



















 

This represents the population at any time . It is complete solution of the model.

**Stability:** For equilibrium states we get from (1)









 and 

provided  or . If  then both roots are complex,  for all , and every solution crashes, hitting zero in finite time. If a solution reaches zero in finite time, we consider the system to have collapsed. If , there are two equilibria: , which increases from to  as  increases from to , and , which decreases from to  as  increases . Thus,  is always unstable and  is always asymptotically stable.

**Q-05:** Describe the Constant harvesting model and analysis the stability of this model.

**Proportional harvesting model:** We consider a population of single species and let be the number of population of the species at any time . If harvesting (reduction of population due to hunting, disease etc.) occurs and some proportion of the population is removed then the harvesting model is defined as

 

where ,  and  are positive constants,  is the linear per capita growth rate,  is the natural carrying capacity and  is the fraction of the population that is harvested in every unit of time.

The equation (1) is a separable or Bernoulli type equation. So separating the variables, we get









Integrating, 

 

where is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get













 

This represents the population at any time . It is complete solution of the model.

The maximum population is





**Stability:** For equilibrium state we get from (1)











The nonzero steady state is

 

This gives a yield

 

If  then  will be negative and so the species will die out. Thus in this case the only realistic steady state is . If  the maximum sustained yield and the new harvesting state from (5) and (4) are



**Q-06:** Describe the Proportional harvesting model and find the condition under which the species will die out.

**Problem**

**Problem-01:** A population  grows accordingly to the Malthus law , where  is a positive constant. Determine how long it takes the population to double in size.

**Solution:** We have





Integrating, 

 

where is an integrating constant.

Suppose the initial population is at , then from (1) we get





Substituting this value in (1) we get

 

Let the population will be double i.e.  at time .

From (2), we get







.

This is the required time.

**Problem-02:** Assume that the population of the Cumilla city increases at a rate proportional to the number of inhabitants at any time. If the population doubles in 40 years in how many years will it be triple?

**Solution:** Let  be the number of inhabitants at any time  and  be the initial population at time . Now according to the question, we have





Integrating, 

 

where is an integrating constant.

Sinceat , then from (1) we get





Substituting this value in (1) we get

 

Again since at , then from (2) we get











Let the population will be triple i.e.  at time .

From (2), we get









years.

This is the required time.

**Problem-03:** Assume that the population of the Cumilla city increases at a rate proportional to the number of inhabitants at any time. If the population was 30,000 in 1970 and 35,000 in 1980. What will be the population in the year 2000 and 1990?

**Solution:** Let  be the number of inhabitants at any time . Now according to the question, we have





Integrating, 

 

where is an integrating constant.

Since at , then from (1) we get

 

Again since at , then from (1) we get

 

Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get

 

Now at the population will be



 (Ans)

Again at the population will be



 (Ans)

**Problem-04:** The world population was estimated to be 1550 millions in 1900 and 2500 millions. Estimate the population at the world in the year 2000 by using Malthusian law.

**Solution:** Let  be the number of population at any time . Now according to the question, we have





Integrating, 

 

where is an integrating constant.

Since at , then from (1) we get

 

Again since at , then from (1) we get

 

Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get

 

Now at the population will be





millions (Ans)

**Problem-05:** Bangladesh population was estimated to be 800 millions in 1980 and 1200 millions in 2000. Estimate the population of Bangladesh in the year 2020 by using Malthusian law.

**Solution:** Let  be the number of population of Bangladesh at any time . Now according to the question, we have





Integrating, 

 

where is an integrating constant.

Since at , then from (1) we get

 

Again since at , then from (1) we get

 

Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get

 

Now at the population will be





millions (Ans)

**Problem-06:** If the population of a country doubles in 50 years in how many years will it triple under the assumption of the Malthusian model? What will be the population in the year 2020?

**Solution:** Let  be the number of population of the country at any time  and  be the initial population at time . Now according to the question, we have





Integrating, 

 

where is an integrating constant.

Since at , then from (1) we get





Substituting this value in (1) we get

 

Again since at , then from (2) we get











Let the population will be triple i.e.  at time .

From (2), we get







years.

This is the required time.

At  the population will be

 (Ans)

**Problem-07:** The population of Cumilla satisfies the logistic law , where the time  is measured in years. If the population of Cumilla was 200000 in 1980, then what will be the population in the year 2000? What would be the possible maximum population of Cumilla?

**Solution:** We have

 

and  

From (1), we have









Integrating, 









 

where is an integrating constant.

Using (2) in (3), we get









Putting this value in (3), we get





 

At the population will be





Hence the population of Cumilla in the year 2000 will be 312965.

The maximum population is









Hence the maximum population of Cumilla will be 1000000.

**Problem-08:** The population (in million)of the USA satisfies the logistic law ,

, , , where the time  is measured in years. Find the maximum population of USA and the population expected in the year 2020.

**Solution:** We have

 

and  

From (1), we have









Integrating, 

 

where is an integrating constant.

Using for in (3), we get

 

From (3) and (4), we get











 

The maximum population is







 

Hence the maximum population of USA will be .

Let  correspond in the years 1930, 1960, 1990 respectively.

Then from (2) and (5), we get



 



 

and 

 

From (8) and (7), we get

 

From (9) and (7), we get

 

Dividing (11) by (10), we get











 

From (10) and (12), we get









The maximum population is

millions

Putting the values of ,  and  in (5), we get

 

This is a formula for all future population.

Now corresponds to the year 2020 and putting in (13), we get



millions

Thus the expected population in the year 2020 is 279.68 millions.

**Problem-09:** The population of a certain city satisfies the logistic law , where the time  is measured in years. If the population of the city was 100,000 in 1980, then determine the population as a function of time for . What will be the population in 2000? In which year will the population be double? How large will the population be in size?

**Solution:** We have



and 

From (1), we have







Integrating, 











where is an integrating constant.

Using (2) in (3), we get









Putting this value in (3), we get







This is the required population as a function of time for .

Now at the population will be





Hence the population of the city in the year 2000 will be 168369.88.

Let the population will be double i.e. for .

Then from (4), we get













years.

The maximum population is









Hence the maximum population of Cumilla will be 1000000.

**Problem-10:** A certain population  obeys the logistic model with specific growth rate  and the carrying capacity . Find the complete solution of the model. Discuss the behavior of the population as . Obtain a formula for the time , when the population size  and the initial population is  with .

**Solution:** We know, the logistic model is



whereis the number of population at any time ,  and  are positive constants and  is very small compared to ,  is carrying capacity.

The equation (1) is a separable or Bernoulli type equation. So separating the variables, we get









Integrating, 



where is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get

















This represents the population at any time . It is complete solution of the model.

When then we have







Therefore we observe that, there is a limit to the growth of as required by biological fact. The maximum population is .

Again, for the time  the population is given by  and the initial population is .

From (3), we have



















This is the required formula for the time .

**Problem-11:** Find the critical points of the harvesting equation , then solve the equation. Also determine the extinction time for the population.

**Solution:** Given that

 

For critical point,













The unique critical point is .

**2nd part:** From (1) we have









Integrating,  

When  then 



Putting the value of  in (2) we get











 

This is the required solution.

**3rd part:** Let at  the population will be extinct i.e. .

From (3) we get









This is the required extinction time.

**Problem-12:** Find the critical points of the harvesting equation , then solve the equation.

**Solution:** Given that

 

For critical point,













The critical point are .

**2nd part:** From (1) we have









Integrating, 

 

When  then 



Putting the value of  in (2) we get









 

This is the required solution.

**Problem-13:** If , , then solve this equation make a realistic comment regarding the solution.

**Solution:** Given that

 

and  

From (1) we have







Integrating, 





 

Using (2) in (3) we get



Putting the value of  in (3) we get

 

This is the required solution.

**Comments:** From (4) we have the following results,

(1).  grows exponentially if .

(2).  decreases and becomes zero if .

**Problem-14:** If  describes the interaction of certain population, then find

(a). a solution of it.

(b). If then find the solution and time of extinction.

**Solution:** Given that

 

**(a).** From (1) we have





Integrating, 





 

This is the required solution.

**(b).** Using  in (2) we get



Putting the value of  in (2) we get

 

This is the required solution.

Again, for time of extinction 





.

This is the required time of extinction.